THE DISCRETE SPECTRUM IN THE SINGULAR FRIEDRICHS MODEL

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Dedicated to M. S. Birman on the occasion of his seventieth birthday

Abstract

A typical result of the paper is the following. Let $\mathbf{H}_\gamma = \mathbf{H}_0 + \gamma \mathbf{V}$ where $\mathbf{H}_0$ is multiplication by $|x|^{2l}$ and $\mathbf{V}$ is an integral operator with kernel $\cos\langle x, y\rangle$ in the space $L_2(\mathbb{R}^d)$. If $l = d/2 + 2k$ for some $k = 0, 1, \ldots$, then the operator $\mathbf{H}_\gamma$ has infinite number of negative eigenvalues for any coupling constant $\gamma \neq 0$. For other values of l, the negative spectrum of $\mathbf{H}_\gamma$ is infinite for $|\gamma| > \sigma_l$ where σ_l is some explicit positive constant. In the case $\pm\gamma \in (0, \sigma_l]$, the number $\mathbf{N}_l^{(\pm)}$ of negative eigenvalues of $\mathbf{H}_\gamma$ is finite and does not depend on γ. We calculate $\mathbf{N}_l^{(\pm)}$.

1. INTRODUCTION

A perturbation of a multiplication operator H_0 by an integral operator V is called a Friedrichs model. It is usually assumed that kernel of V is a Hölder continuous (with an exponent larger than 1/2) function $\mathbf{v}(x, y)$ of one-dimensional variables x and y which decays sufficiently rapidly as $|x| + |y| \to \infty$. Then (see [5]) the wave operators for the pair H_0, $H_1 = H_0 + V$ exist and are complete. Moreover, the operator H_1 does not have the singular continuous spectrum, and its discrete spectrum is finite. It is important that the results of [5] are applicable to the case when a kernel $\mathbf{v}(x, y)$ is itself a compact operator in an auxiliary Hilbert space.

A somewhat different situation was considered in the paper [4] where the operator $\mathbf{H}_0$ of multiplication by $|x|^{2l}$, $l > 0$, in the space $L_2(\mathbb{R}^d)$ was perturbed by an integral operator of Fourier type. More precisely, the perturbation was defined by one of the equalities

$$(\mathbf{V}^{(c)} f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \cos\langle x, y\rangle f(y) dy \quad \text{or} \quad (\mathbf{V}^{(s)} f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sin\langle x, y\rangle f(y) dy \tag{1.1}$$

and $\mathbf{H}_\gamma^{(c)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(c)}$ or $\mathbf{H}_\gamma^{(s)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(s)}$ where $\gamma \in \mathbb{R}$ is a coupling constant. An interesting feature of this model is that $\mathbf{V}^{(c)}$ and $\mathbf{V}^{(s)}$ are invariant with respect to the Fourier transform so that one could have chosen $\mathbf{H}_0 = (-\Delta)^l$ for the "unperturbed" operator. Passing to the spherical coordinates and considering the space $L_2(\mathbb{R}^d)$ as $L_2(\mathbb{R}_+; L_2(\mathbb{S}^{d-1}))$, we can fit the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$ into the framework of the Friedrichs model. However, since the

1991 Mathematical Subject Classification: Primary 35J10, 47A75; Secondary 81U20

kernels $\cos\langle x,y\rangle$ or $\sin\langle x,y\rangle$ do not tend to 0 as $|x|\to\infty$ or (and) $|y|\to\infty$, the results of the paper [5] are not applicable to perturbations (1.1) (even in the case $d=1$).

Due to oscillations of its kernel, the operators (1.1) are bounded in the space $L_2(\mathbb{R}^d)$ but they are not compact, even relatively with respect to $\mathbf{H}_0$. Nevertheless, as shown in [4], the essential spectra of the operators $\mathbf{H}_\gamma^{(c)}$, $\mathbf{H}_\gamma^{(s)}$ and $\mathbf{H}_0$ coincide. This implies that the negative spectra of the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$ consist of eigenvalues of finite multiplicity which may accumulate at the bottom of the essential spectrum (point zero) only. Moreover, in the case $2l>d$, the trace-class technique was used in [4] to prove, for the pairs $\mathbf{H}_0$, $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_0$, $\mathbf{H}_\gamma^{(s)}$, the existence of the wave operators and their completeness.

Our goal here is to study the discrete spectrum of the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$. The space $L_2(\mathbb{R}^d)$ decomposes into the orthogonal sum of subspaces $\mathfrak{H}_n$, $n=0,1,2,\dots$ constructed in terms of the spherical functions of order n and invariant with respect to the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$. On the subspaces $\mathfrak{H}_n$, these operators reduce to operators H_γ acting in the space $L_2(\mathbb{R}_+)$. Operators H_γ have a form $H_\gamma = H_0+\gamma V$ where H_0 is again multiplication by x^{2l}, $l>0$, and

$$(Vf)(x)=\int_0^\infty v(xy)f(y)dy. \tag{1.2}$$

The function v depends of course on n and on the index "c" or "s" but is always expressed in terms of a Bessel function.

Therefore, we consider first operators $H_\gamma = H_0+\gamma V$ in the space $L_2(\mathbb{R}_+)$ where V is given by (1.2) with a rather arbitrary real function v. We suppose that $v(t)$ has a sufficiently regular behaviour as $t\to 0$ and $t\to\infty$. Since operators (1.2) never satisfy conditions of [5], this version of the Friedrichs model is called singular here. It turns out that the negative spectrum of the operator H_γ is very sensitive with respect to the coupling constant γ and the parameter l. Thus, if the asymptotic expansion of $v(t)$ as $t\to 0$ contains the term vt^r, then, in the case $l=r+1/2$, the operator H_γ has infinite number of negative eigenvalues for any coupling constant $\gamma\neq 0$. For other values of l, the negative spectrum of H_γ is infinite for $|\gamma|>\sigma_l$ where σ_l is some explicit positive constant. In the case $\pm\gamma\in(0,\sigma_l]$, the number $N_l^{(\pm)}$ of negative eigenvalues of H_γ does not depend on γ. We calculate $N_l^{(\pm)}$ in terms of the asymptotic expansion of the function $v(t)$ as $t\to 0$. We emphasize that both positive and negative parts of perturbation (1.2) are, in general, non-trivial, and our results on the discrete spectrum of H_γ take into account their "interaction".

The results on the discrete spectrum in the singular Friedrichs model can be compared with those for the Schrödinger operator $-\Delta+\gamma q(x)$ whose potential $q(x)$ has a critical decay at infinity. Suppose, for example, that $x\in\mathbb{R}$ and a negative function $q(x)$ has the asymptotics $q(x)\sim -|x|^{-2}$ as $|x|\to\infty$. Then for sufficiently small $\gamma>0$ the negative spectrum of the operator $-\Delta+\gamma q(x)$ consists of exactly one eigenvalue, it is finite for $\gamma\in(0,1/4)$ and it is infinite for $\gamma>1/4$. Note also that the mechanism for appearance of infinite number of negative eigenvalues in the singular Friedrichs model resembles the same phenomena (the Efimov's effect, see[6]) for the three-particle Schrödinger operator with short-range pair potentials.

Our calculation of the number of negative eigenvalues relies on the Birman-Schwinger principle. We need a presentation of this theory somewhat different from the original paper

[2] and adapted to perturbations without a definite sign. This is done in Section 2 in the abstract framework. The negative spectrum of the operator H_γ in the space $L_2(\mathbb{R}_+)$ is studied in Section 3. In Section 4, we use these results to calculate the total number of negative eigenvalues of the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$ in the space $L_2(\mathbb{R}^d)$.

2. THE BIRMAN-SCHWINGER PRINCIPLE

1. Let H_0 be an arbitrary self-adjoint positive operator with domain $\mathcal{D}(H_0)$ in a Hilbert space $\mathcal{H}$. Following [2], we define the "full" Hamiltonian $H = H_0 + V$ by means of the corresponding quadratic form. We suppose that the real quadratic form $v[f,f]$ of the perturbation V is defined for $f \in \mathcal{D}(H_0^{1/2})$ and is compact in the Hilbert space $\mathcal{D}(H_0^{1/2})$ with the scalar product $(f_1, f_2)_{H_0^{1/2}} = (H_0^{1/2} f_1, H_0^{1/2} f_2) + (f_1, f_2)$. This means that $|v[f,f]| \leq C ||f||^2_{H_0^{1/2}}$ and the operator T defined by the relation

$$(Tu, u) = v[(H_0 + I)^{-1/2} u, (H_0 + I)^{-1/2} u], \quad \forall u \in \mathcal{H}, \tag{2.1}$$

is compact in the space $\mathcal{H}$. Under this assumption the form

$$h[f,f] = ||H_0^{1/2} f||^2 + v[f,f]$$

is closed on $\mathcal{D}(H_0^{1/2})$. So there exists (see [3]) a unique self-adjoint, semi-bounded from below, operator H such that $\mathcal{D}(|H|^{1/2}) = \mathcal{D}(H_0^{1/2})$ and $h[f,g] = (Hf, g)$ for any $f \in D(H)$ and any $g \in D(H_0^{1/2})$.

The resolvent identity for the operators H_0, H can be written as

$$(H + c)^{-1} - (H_0 + c)^{-1} = -(H_0 + c)^{-1}(H_0 + I)^{1/2} T (H_0 + I)^{1/2} (H + c)^{-1}, \tag{2.2}$$

where c is a sufficiently large positive constant. Since the operator $(H_0 + I)^{1/2}(H + c)^{-1}$ is bounded, the right-hand side of (2.2) is a compact operator, and hence by Weyl's theorem, the essential spectra σ_{ess} of the operators H_0 and H coincide. This result was established in [2].

Our goal here is to calculate the total multiplicity

$$N = \dim E_H(-\infty, 0) \tag{2.3}$$

of the negative spectrum of the operator H. To that end we introduce some auxiliary objects. Let us consider the set $\mathcal{R}$ of elements $u = H_0^{1/2} f$ for all $f \in \mathcal{D}(H_0^{1/2})$. The set $\mathcal{R}$ is endowed with the norm

$$||u||^2_{\mathcal{R}} = ||u||^2 + ||H_0^{-1/2} u||^2 \tag{2.4}$$

and is dense in $\mathcal{H}$ since $\operatorname{Ker} H_0^{1/2} = \{0\}$. Let us define the bounded quadratic form a on $\mathcal{R}$ by the relation

$$a[u,u] = v[H_0^{-1/2} u, H_0^{-1/2} u], \quad u \in \mathcal{R}. \tag{2.5}$$

Lemma 2.1 *The number* (2.3) *equals the maximal dimension of subspaces* $\mathcal{L} \subset \mathcal{R}$ *such that*

$$a[u,u] < -||u||^2, \quad \forall u \in \mathcal{L}. \tag{2.6}$$

Proof. – By the spectral theorem, N equals the maximal dimension of subspaces $\mathcal{K} \subset \mathcal{D}(H_0^{1/2})$ such that

$$h[f,f] < 0, \quad \forall f \in \mathcal{K}. \tag{2.7}$$

Setting $u = H_0^{1/2} f$ and taking into account definition (2.5), we see that (2.6) and (2.7) are equivalent if $\mathcal{L} = H_0^{1/2}\mathcal{K}$. Moreover, $\dim \mathcal{K} = \dim \mathcal{L}$ since $\operatorname{Ker} H_0^{1/2} = \{0\}$. $\square$

Lemma 2.2 *Let B be any self-adjoint operator and let $\mathfrak{M} \subset \mathcal{D}(B)$ be some set dense in $\mathcal{D}(B)$ in the B-metrics defined by $||u||_B^2 = ||Bu||^2 + ||u||^2$. Then the total multiplicity* $\dim E_B(-\infty, 0)$ *of the negative spectrum of the operator B equals the maximal dimension of subspaces $\mathcal{L} \subset \mathfrak{M}$ such that $(Bu,u) < 0$, $\forall u \in \mathcal{L}$.*

In the case of a bounded operator B, this assertion is Lemma 1.2 from [2]. In the general case, its proof is practically the same.

2. In terms of the form (2.5), a simple version of the Birman-Schwinger principle can be formulated as follows.

Proposition 2.3 *Suppose that*

$$a[u,u] = (Au,u) \tag{2.8}$$

for some bounded self-adjoint operator A in the space $\mathcal{H}$ and all $u \in \mathcal{R}$. Let

$$M = \dim E_A(-\infty, -1) \tag{2.9}$$

be the total multiplicity of the spectrum of the operator A in the interval $(-\infty, -1)$. *Then the numbers* (2.3) *and* (2.9) *are equal, that is $N = M$.*

Proof. – Let n be the maximal dimension of subspaces $\mathcal{L}_0 \subset \mathcal{R}$ such that

$$(Au_0, u_0) < -||u_0||^2, \quad \forall u_0 \in \mathcal{L}_0. \tag{2.10}$$

It follows from (2.8) and Lemma 2.1 that $N = n$. Since $\mathcal{R}$ is dense in $\mathcal{H}$, the equality $M = n$ follows from Lemma 2.2. $\square$

This result is contained in the paper [2] because, under assumptions of Proposition 2.3, the form $v[f,f]$ is bounded in the space $\mathfrak{H}$ with the norm $||f||_{\mathfrak{H}} = ||H_0^{1/2} f||$ and consequently is closable in this space. A more difficult case when the form $v[f,f]$ is not closable in $\mathfrak{H}$ was also investigated in [2]. Our aim here is to reconsider this situation and to formulate results in terms of the form (2.5). This is convenient in the case when the form $v[f,f]$ takes values of both signs. Actually, we suppose that equality (2.8) holds only up to some finite number of squares of (unbounded) functionals $\varphi_1, \dots, \varphi_n$, that is

$$a[u,u] = (Au,u) - \sum_{j=1}^{m} |\varphi_j(u)|^2 + \sum_{j=m+1}^{n} |\varphi_j(u)|^2. \tag{2.11}$$

Of course, one or both sums in (2.11) may be absent, that is we do not exclude the cases $m = 0$ or (and) $n = m$.

Let us give first sufficient conditions for the negative spectrum of the operator H to be infinite. In this case we do not assume that A is bounded and replace equality (2.11) by an estimate.

Theorem 2.4 *Let A be a self-adjoint operator with domain $\mathcal{D}(A)$. Suppose that a linear set $\mathcal{R}_0 \subset \mathcal{D}(A) \cap \mathcal{R}$ is dense in $\mathcal{D}(A)$ in the A-metrics. Assume that for all $u \in \mathcal{R}_0$ the form* (2.5) *satisfies the estimate*

$$a[u,u] \leq (Au,u) + \sum_{j=1}^{p} |\varphi_j(u)|^2, \tag{2.12}$$

where $\varphi_1, \dots, \varphi_p$ is a system of linear functionals defined on $\mathcal{R}_0$. Then the negative spectrum of the operator H is infinite provided $\dim E_A(-\infty, -1) = \infty$.

Proof. – By Lemma 2.2, for any k, there exists a subspace $\mathcal{L}_0 \subset \mathcal{R}_0$ such that $\dim \mathcal{L}_0 = k$ and inequality (2.10) holds. Let $\mathcal{L} \subset \mathcal{L}_0$ consist of elements $u \in \mathcal{L}_0$ such that $\varphi_j(u) = 0$ for all $j = 1, \dots, p$. Clearly, $\dim \mathcal{L} \geq k - p$. It follows from (2.12) that $a[u,u] \leq (Au,u)$ for $u \in \mathcal{L}$. Therefore (2.10) implies (2.6). So, by Lemma 2.1, the negative spectrum of H contains at least $k - p$ eigenvalues. It remains to take into account that k is arbitrary. $\square$

3. To calculate the number of negative eigenvalues of the operator H, we need several auxiliary assertions. The first of them has a purely algebraic nature and is quite standard.

Lemma 2.5 *Let $\mathfrak{R}$ be some linear space (perhaps of infinite dimension) and let $\varphi_1, \dots, \varphi_n, \varphi : \mathfrak{R} \to \mathbb{C}$ be linear functionals. Suppose that $\varphi_1, \dots, \varphi_n$ are linearly independent and denote by $\mathcal{N}$ the set of their common zeros, i.e. $u \in \mathcal{N}$ iff $\varphi_j(u) = 0$ for any $j = 1, \dots, n$. Assume that $\varphi(u) = 0$ for all $u \in \mathcal{N}$. Then $\varphi = \sum_{j=1}^{n} \alpha_j \varphi_j$ for some $\alpha_j \in \mathbb{C}$.*

To exclude the case when the sums in (2.11) are degenerate, we introduce the following

Definition 2.6 *Let $\mathfrak{R} \subset \mathcal{H}$ be a linear set dense in $\mathcal{H}$ and let $\varphi_1, \dots, \varphi_n$ be linear functionals defined on $\mathfrak{R}$. We call $\varphi_1, \dots, \varphi_n$ strongly linear independent if the inequality*

$$|\sum_{j=1}^{n} \alpha_j \varphi_j(u)| \leq C||u||, \quad \forall u \in \mathfrak{R}, \tag{2.13}$$

(here C is some positive constant) implies that $\alpha_j = 0$ for all $j = 1, \dots, n$.

Thus, it is impossible to find a linear combination of strongly linear independent functionals which is a bounded functional on $\mathcal{H}$. Of course, the strong linear independence ensures the usual linear independence, and every functional from a strongly linear independent system is unbounded.

Lemma 2.7 *If a functional φ (defined on a dense set $\mathfrak{R}$) is unbounded, then the set of its zeros is dense.*

Proof. – There exists a sequence $w_k \in \mathfrak{R}$ such that $\varphi(w_k) = 1$ and $||w_k|| \to 0$ as $k \to \infty$. Moreover, for any given sequence $\eta_k \to 0$, choosing a subsequence of w_k, we can satisfy the bound $||w_k|| \leq \eta_k$. Let $h \in \mathcal{H}$ be an arbitrary element. Then there exists a sequence $u_k \in \mathfrak{R}$ such that $u_k \to h$ in $\mathcal{H}$ as $k \to \infty$. Put

$$u_k^{(0)} = u_k - \varphi(u_k)w_k.$$

Clearly, $\varphi(u_k^{(0)}) = 0$ and $u_k^{(0)} \to h$ provided $|\varphi(u_k)|\,||w_k|| \to 0$ as $k \to \infty$. $\square$

Corollary 2.8 *The set of common zeros of any finite system of unbounded functionals (in particular, of a strongly independent system) is dense in* $\mathcal{H}$.

Let us discuss some properties of functionals satisfying Definition 2.6.

Lemma 2.9 *Let* $\varphi_1, \dots, \varphi_n$ *be strongly linear independent functionals and let* $\mathcal{N}_m$ *be the set of common zeros of functionals* $\varphi_j, j = 1, \dots, n,\ j \neq m$. *Then the restriction of* φ_m *on* $\mathcal{N}_m$ *is unbounded.*

Proof. – In the opposite case, there exists a bounded functional φ such that $\varphi_m(u) = \varphi(u)$ for $u \in \mathcal{N}_m$. So it follows from Lemma 2.5 that $\varphi(u) = \varphi_m(u) + \sum_{j \neq m} \alpha_j \varphi_j(u)$ for all $u \in \mathfrak{R}$. This contradicts the strong linear independence of $\varphi_1, \dots, \varphi_n$. $\square$

Lemma 2.10 *Let* $\varphi_1, \dots, \varphi_n$ *be strongly linear independent functionals and let* $\mathcal{L}_0$ *be a finite-dimensional subspace of* $\mathfrak{R}$. *Then for any* $\lambda > 0$ *and any* $m = 1, \dots, n$ *there exists a vector* $u^{(m)}, ||u^{(m)}|| = 1$, *such that* $u^{(m)} \notin \mathcal{L}_0$, $\varphi_j(u^{(m)}) = 0$ *for* $j = 1, \dots, n, j \neq m$, *and* $|\varphi_m(u^{(m)})| \geq \lambda$.

Proof. – By Lemma 2.9, there exists a sequence z_k such that $||z_k|| = 1$, $|\varphi_m(z_k)| \to \infty$ and $\varphi_j(z_k) = 0$ for all $j = 1, \dots, n,\ j \neq m$. Since $\dim \mathcal{L}_0 < \infty$, we have that

$$\sup_{z \in \mathcal{L}_0,\, ||z||=1} |\varphi_m(z)| < \infty.$$

Therefore $z_k \notin \mathcal{L}_0$ and $|\varphi_m(z_k)| \geq \lambda$ for k large enough. So we can set $u^{(m)} = z_k$ for such k. $\square$

This assertion can be generalized.

Lemma 2.11 *Under assumptions of Lemma* 2.10, *for any* $\lambda > 0$ *and any* $m \leq n$, *there exists a set of normalized vectors* $u_i \notin \mathcal{L}_0$, $i = 1, \dots, m$, *such that*

$$\varphi_j(u_i) = 0, \quad j = 1, \dots, n,\ j \neq i, \tag{2.14}$$

and

$$|\varphi_i(u_i)| \geq \lambda. \tag{2.15}$$

Moreover, for the subspace $\mathcal{L} \subset \mathfrak{R}$ *spanned by* $\mathcal{L}_0$ *and* $u_1, \dots, u_m$

$$\dim \mathcal{L} = \dim \mathcal{L}_0 + m. \tag{2.16}$$

Proof. – In the case $m = 1$ Lemmas 2.10 and 2.11 councide. Suppose that we have already constructed $u_1, \dots, u_{m-1}$ such that relations (2.14), (2.15) are satisfied for $i = 1, \dots, m-1$ and the subspace $\tilde{\mathcal{L}}$ spanned by $\mathcal{L}_0$ and $u_1, \dots, u_{m-1}$ has dimension $\dim \mathcal{L}_0 + m - 1$. Then existence of the vector u_m with all necessary properties follows again from Lemma 2.10 applied to the subspace $\tilde{\mathcal{L}}$. $\square$

4. In Definition 2.6, $\mathfrak{R}$ is an arbitrary linear set but, if $\mathfrak{R}$ is a Banach space, we suppose that the functionals $\varphi_1, \dots, \varphi_n$ are bounded on this space (but not on $\mathcal{H}$, of course). In our study of the negative spectrum of the operator H, the role of $\mathfrak{R}$ is played by the space $\mathcal{R} = H_0^{1/2}\mathcal{D}(H_0^{1/2})$ endowed with the norm (2.4). Now we are in position to formulate the main result of this section.

Theorem 2.12 *Let $\varphi_1, \dots, \varphi_n$ be strongly linear independent functionals defined on $\mathcal{R}$ and let A be a bounded self-adjoint operator. Assume that equality* (2.11) *holds for all $u \in \mathcal{R}$. Then numbers* (2.3) *and* (2.9) *are related by the equality*

$$N = M + m. \tag{2.17}$$

Proof. – First, we check that $N \leq M + m$. By Lemma 2.1, there exists a subspace $\mathcal{L} \subset \mathcal{R}$ such that (2.6) is fulfilled and $\dim \mathcal{L} = N$ ($\dim \mathcal{L}$ is an arbitrary large number if $N = \infty$). It follows from (2.11) that (2.10) is satisfied for

$$\mathcal{L}_0 = \{u \in \mathcal{L} : \varphi_j(u) = 0,\ j = 1, \dots, m\}.$$

Clearly, $\dim \mathcal{L}_0 \geq \dim \mathcal{L} - m$. Since $M \geq \dim \mathcal{L}_0$, this implies that $M \geq N - m$.

It remains to check that $N \geq M + m$. Let the set $\mathcal{N} \subset \mathcal{R}$ be defined by the condition: $u \in \mathcal{N}$ iff $\varphi_j(u) = 0$ for all $j = 1, \dots, n$. By Corollary 2.8, this set is dense in $\mathcal{H}$. Therefore, by Lemma 2.2, M equals the maximal dimension of $\mathcal{L}_0 \subset \mathcal{N}$ where (2.10) holds. Let $\mathcal{L}$ be the subspace constructed in Lemma 2.11 for $\mathfrak{R} = \mathcal{R}$ and sufficiently large λ which will be chosen later. According to (2.16), we need only to verify relation (2.6) on this subspace. Every vector $u \in \mathcal{L}$ has the form

$$u = u_0 + \sum_{i=1}^{m} \beta_i u_i, \quad \text{where} \quad u_0 \in \mathcal{L}_0, \quad \beta_j \in \mathbb{C}, \tag{2.18}$$

so that, for $B = A + I$,

$$(Bu, u) = (Bu_0, u_0) + 2\,\mathrm{Re} \sum_{i=1}^{m} \bar{\beta}_i (Bu_0, u_i) + \sum_{i,j=1}^{m} \beta_i \bar{\beta}_j (Bu_i, u_j). \tag{2.19}$$

Recall that $\mathcal{L}_0 \subset \mathcal{N}$ and, consequently, $\varphi_j(u_0) = 0$ for all $j = 1, \dots, n$. Therefore it follows from (2.14) and (2.18) that

$$\varphi_i(u) = \beta_i \varphi_i(u_i), \quad i = 1, \dots, m, \quad \text{and} \quad \varphi_j(u) = 0, \quad j = m+1, \dots, n. \tag{2.20}$$

Comparing (2.19) and (2.20), we obtain the bound for the form (2.11):

$$a[u,u] + ||u||^2 = (Bu, u) - \sum_{i=1}^{m} |\varphi_i(u)|^2$$

$$\leq (Bu_0, u_0) + 2||Bu_0|| \sum_{i=1}^{m} |\beta_i| + mb \sum_{i=1}^{m} |\beta_i|^2 - \sum_{i=1}^{m} |\beta_i|^2 |\varphi_i(u_i)|^2,$$

where $b = \max_{1 \le i,j \le m} |(Bu_i, u_j)|$.

Thus, for the proof of (2.6), it suffices to check that for all $u_0 \in \mathcal{L}_0$ and any numbers β_i

$$(mb - |\varphi_i(u_i)|^2)|\beta_i|^2 + 2||Bu_0||\,|\beta_i| + m^{-1}(Bu_0, u_0) < 0, \quad i = 1, \dots, m. \tag{2.21}$$

Since $(Bu_0, u_0) < 0$ and the dimension of $\mathcal{L}_0$ is finite, we have that

$$(Bu_0, u_0) \le -b_0||u_0||^2 \quad \text{and} \quad ||Bu_0|| \le b_1||u_0||$$

for some $b_0, b_1 > 0$. Taking also into account (2.15), we see that (2.21) is satisfied if

$$(\lambda^2 - mb)|\beta_i|^2 - 2b_1||u_0||\,|\beta_i| + m^{-1}b_0||u_0||^2 > 0.$$

The last inequality holds for arbitrary β_i if λ is large enough, that is $\lambda^2 > m(b + b_1^2 b_0^{-1})$. This concludes the proofs of (2.6) and hence of Theorem 2.12. $\square$

5. It is easy to extend Proposition 2.3 and Theorem 2.12 to the case of unbounded operators A. We formulate such results but do not use them in the sequel.

Proposition 2.3 bis *Let A be a self-adjoint operator in the space $\mathcal{H}$ and let $\mathcal{R}_0 \subset \mathcal{R} \cap \mathcal{D}(A)$ be a linear set dense in $\mathcal{R}$ in the $\mathcal{R}$-metrics and dense in $\mathcal{D}(A)$ in the A-metrics. If equality* (2.8) *holds for all $u \in \mathcal{R}_0$, then $N = M$.*

Theorem 2.12 bis *Let A be a self-adjoint operator in the space $\mathcal{H}$ and let $\mathcal{R}_0 \subset \mathcal{R} \cap \mathcal{D}(A)$ be a linear set dense in $\mathcal{R}$ in the $\mathcal{R}$-metrics and dense in $\mathcal{D}(A)$ in the A-metrics. Assume that functionals $\varphi_1, \dots, \varphi_n$ are strongly linear independent in the Hilbert space $\mathcal{D}(A)$ (that is $||u||$ in the right-hand of* (2.13) *is replaced by $||u||_A$). Then equality* (2.17) *is fulfilled.*

Proofs of Proposition 2.3 bis and Theorem 2.12 bis are practically the same as those of Proposition 2.3 and Theorem 2.12.

3. THE FRIEDRICHS MODEL

1. Our study of the discrete spectrum in the Friedrichs model relies on the Mellin transform $\mathbf{M}$ defined by the equality

$$(\mathbf{M}u)(\lambda) = (2\pi)^{-1/2} \int_0^\infty x^{-1/2 - i\lambda} u(x)dx. \tag{3.1}$$

The operator $\mathbf{M} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$ is unitary.

Lemma 3.1 *Suppose that a function $b(t)$ is locally bounded on $(0, \infty)$ and the integral*

$$\int_0^\infty b(t)t^{-1/2-i\lambda}dt =: \beta(\lambda)$$

converges at $t = 0$ and $t = \infty$ uniformly in $\lambda \in \mathbb{R}$. Then for any function $u \in C_0^\infty(\mathbb{R}_+)$

$$\int_0^\infty \int_0^\infty b(xy)u(y)\overline{u(x)}dxdy = \int_{-\infty}^\infty \beta(\lambda)(\mathbf{M}u)(-\lambda)\overline{(\mathbf{M}u)(\lambda)}d\lambda. \tag{3.2}$$

Proof. – Changing in the left-hand side the variables $x = e^t$, $y = e^s$ and denoting $u_1(t) = e^{t/2}u(e^t)$, $b_1(t) = e^{t/2}b(e^t)$, we rewrite it as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_1(t+s)u_1(s)\overline{u_1(t)}dtds.$$

If $u_1(t) = 0$ for $|t| \geq n$, then, by virtue of the convolution formula and the Parseval identity, this integral equals

$$\int_{-\infty}^{\infty} \beta_n(\lambda)\hat{u}_1(-\lambda)\overline{\hat{u}_1(\lambda)}d\lambda, \tag{3.3}$$

where $\hat{u}_1$ is the Fourier transform of u_1 and

$$\beta_n(\lambda) = \int_{-2n}^{2n} b_1(t)e^{-i\lambda t}dt.$$

Under our assumptions functions $\beta_n(\lambda)$ are uniformly bounded and converge to $\beta(\lambda)$ as $n \to \infty$. Since $\hat{u}_1 = \mathbf{M}u$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$, we can pass to the limit $n \to \infty$ in integral (3.3). The expression obtained equals the right-hand side of (3.2). $\square$

Under assumptions of Lemma 3.1, the function $\beta(\lambda)$ is of course continuous and bounded.

Let us define a unitary mapping $U : L_2(\mathbb{R}) \to L_2(\mathbb{R}_+; \mathbb{C}^2)$ and a 2×2- matrix $\mathcal{B}(\lambda)$ by the equalities

$$(Uw)(\lambda) = \begin{pmatrix} w(\lambda) \\ w(-\lambda) \end{pmatrix}, \quad \mathcal{B}(\lambda) = \begin{pmatrix} 0 & \beta(\lambda) \\ \overline{\beta(\lambda)} & 0 \end{pmatrix}, \quad \lambda > 0. \tag{3.4}$$

If $\beta(-\lambda) = \overline{\beta(\lambda)}$, then

$$\int_{-\infty}^{\infty} \beta(\lambda)w(-\lambda)\overline{w(\lambda)}d\lambda = (\mathcal{B}Uw, Uw)_{L_2(\mathbb{R}_+;\mathbb{C}^2)}, \tag{3.5}$$

where $\mathcal{B}$ is the operator of multiplication by $\mathcal{B}(\lambda)$. Since eigenvalues of the matrix $\mathcal{B}(\lambda)$ equal $\pm|\beta(\lambda)|$, we have

Lemma 3.2 *Let $\beta(\lambda)$ be a continuous function of $\lambda \in (0, \infty)$ and*

$$p = \max_{\lambda\in\mathbb{R}_+} |\beta(\lambda)|, \quad q = \min_{\lambda\in\mathbb{R}_+} |\beta(\lambda)|$$

(*the case $p = \infty$ is not excluded*). *Then the spectrum of $\mathcal{B}$ consists of the union $[-p, -q] \cup [q, p]$.*

2. Let now $\mathcal{H} = L_2(\mathbb{R}_+)$, let H_0 be multiplication by the function x^{2l}, $l > 0$, and let an integral operator V be defined by formula (1.2). We suppose that the function $v(t) = \overline{v(t)}$ is locally bounded on $(0, \infty)$. Our assumption on its behaviour as $t \to \infty$ will be made in terms of the Mellin transform.

Assumption 3.3 *The integral*

$$\int_1^{\infty} v(t)t^{-1/2-l-i\lambda}dt$$

converges uniformly (*but perhaps not absolutely*) *in $\lambda \in \mathbb{R}$.*

The precise definition of the operator $H_\gamma = H_0 + \gamma V$ where a coupling constant $\gamma \in \mathbb{R}$ can be given on the basis of the following

Lemma 3.4 *Let Assumption* 3.3 *hold and let* $v(t) = O(t^r)$ *with* $r > -1/2$ *as* $t \to 0$. *Then the operator* $T = (H_0 + I)^{-1/2} V (H_0 + I)^{-1/2}$ *is compact.*

Proof. – Let χ_R be the characteristic function of the interval $(0, R)$ and $\tilde{\chi}_R = 1 - \chi_R$. Denote by V_R and $\tilde{V}_R$ the integral operators with kernels $v(xy)\chi_R(xy)$ and $v(xy)\tilde{\chi}_R(xy)$, respectively. First, we check that the operator $(H_0 + I)^{-1/2} V_R (H_0 + I)^{-1/2}$ is compact for any $R > 0$. Indeed, let us consider the operator-function

$$F(z) = (H_0 + I)^{-z} V_R (H_0 + I)^{-z}, \quad \operatorname{Re} z \geq 0.$$

The function $v(t)\chi_R(t)$ satisfies the assumptions of Lemma 3.1 so that V_R is a bounded operator. Consequently, the operators $F(z)$ are bounded for all $\operatorname{Re} z \geq 0$, and the function $F(z)$ is analytic for $\operatorname{Re} z > 0$ and is continuous in z up to the line $\operatorname{Re} z = 0$. On the other hand, $F(z)$ is the integral operator with kernel

$$(x^{2l} + 1)^{-z} v(xy) \chi_R(xy) (y^{2l} + 1)^{-z}.$$

So $F(z)$ belongs to the Hilbert-Schmidt class if $4l \operatorname{Re} z > 1$. By complex interpolation, this implies that $F(z)$ is compact for all $\operatorname{Re} z > 0$.

To finish the proof, it suffices to show that the norm of the operator $B_R = H_0^{-1/2} \tilde{V}_R H_0^{-1/2}$ tends to zero as $R \to \infty$. Clearly, B_R is also the integral operator with kernel $b_R(xy) = (xy)^{-l} v(xy) \tilde{\chi}_R(xy)$. By virtue of Assumption 3.3, we can apply Lemmas 3.1 and 3.2 to it. This implies that

$$||B_R|| = \max_{\lambda} \left| \int_R^\infty v(t) t^{-1/2-l-i\lambda} dt \right| \to 0 \quad \text{as} \quad R \to \infty. \quad \square$$

Thus, equality (2.1) holds with a compact operator T and hence the operator $H_\gamma = H_0 + \gamma V$ can be defined as a self-adjoint operator in terms of the corresponding quadratic form. Moreover, H_γ is semi-bounded from below and $\sigma_{ess}(H_\gamma) = [0, \infty)$ for any $\gamma \in \mathbb{R}$.

To study the discrete spectrum of H_γ, we need additional conditions on $v(t)$ as $t \to 0$.

Assumption 3.5 *Suppose that, as* $t \to 0$,

$$v(t) = \sum_{k=1}^{N} v_k t^{r_k} + O(t^{r_{N+1}}), \tag{3.6}$$

where $-1/2 < r_1 < \cdots < r_N < r_{N+1}$ *and* $l < r_{N+1} + 1/2$.

We assume that $v_k \neq 0$ for $k = 1, \ldots, N$ but do not exclude the case when the sum in (3.6) is absent, that is

$$v(t) = O(t^{r_1}) \quad \text{with} \quad r_1 > l - 1/2. \tag{3.7}$$

We set

$$b_l(t) = t^{-l}(v(t) - \sum_{r_k < l - 1/2} v_k t^{r_k}) \tag{3.8}$$

($b_l(t) = t^{-l} v(t)$ in the case (3.7)) and

$$\beta_l^{(\kappa)}(\lambda) = \int_0^\infty \chi^{(\kappa)}(t) b_l(t) t^{-1/2-i\lambda} dt, \quad \kappa = 0, 1, \tag{3.9}$$

where $\chi^{(0)}(t)$ and $\chi^{(1)}(t)$ are the characteristic functions of the intervals $(0,1)$ and $(1,\infty)$, respectively. In definition (3.9) of the function $\beta_l^{(0)}$ we suppose that $l \neq r_n + 1/2$ for $n = 1, \dots, N$.

The following assertions are quite elementary.

Lemma 3.6 *Let Assumption* 3.3 *be satisfied. Then the integral* (3.9) *for* $\kappa = 1$ *converges at* $t = \infty$ *uniformly in* $\lambda \in \mathbb{R}$.

Lemma 3.7 *Let Assumption* 3.5 *be satisfied. Suppose that* $l \neq r_n + 1/2$ *for* $n = 1, \dots, N$. *Then the integral* (3.9) *for* $\kappa = 0$ *converges at* $t = 0$ *uniformly in* $\lambda \in \mathbb{R}$.

If conditions of both Lemmas 3.6 and 3.7 are fulfilled, then the function

$$\beta_l(\lambda) = \int_0^\infty b_l(t) t^{-1/2 - i\lambda} dt \tag{3.10}$$

is continuous and

$$p_l = \max_{\lambda \in \mathbb{R}} |\beta_l(\lambda)| < \infty.$$

Let us denote by $N_l^{(\pm)}$ the number of $k = 1, \dots, N$ such that $r_k < l - 1/2$ and $\mp v_k > 0$. In the case (3.7) we set $N_l^{(\pm)} = 0$. The main result of this section is formulated in the following

Theorem 3.8 *Let Assumptions* 3.3 *and* 3.5 *be satisfied.*

1^0 *Suppose that* $l \neq r_n + 1/2$ *for* $n = 1, \dots, N$. *Put* $\sigma_l = p_l^{-1}$. *Then the negative spectrum of the operator* H_γ *is infinite if* $|\gamma| > \sigma_l$. *In the case* $\pm\gamma \in (0, \sigma_l]$, *it consists of* $N_l^{(\pm)}$ *eigenvalues.*

2^0 *If* $l = r_n + 1/2$ *for some* $n = 1, \dots, N$, *then the negative spectrum of the operator* H_γ *is infinite for any* $\gamma \neq 0$.

We start the proof with calculating the quadratic form (2.5), which equals now

$$a_l[u,u] = \gamma \int_0^\infty \Big(\int_0^\infty v(xy) y^{-l} u(y) dy \Big) x^{-l} \overline{u(x)} dx. \tag{3.11}$$

Using notation (3.8), we can rewrite (3.11) for $u \in C_0^\infty(\mathbb{R}_+)$ as

$$a_l[u,u] = \gamma b_l[u,u] + \gamma \sum_{r_k < l - 1/2} v_k |\Phi_{r_k - l}(u)|^2, \tag{3.12}$$

where

$$b_l[u,u] = \int_0^\infty \int_0^\infty b_l(xy) u(y) \overline{u(x)} dx dy,$$

and

$$\Phi_p(u) = \int_0^\infty x^p u(x) dx. \tag{3.13}$$

Of course, in the case (3.7) the sum in (3.12) is absent. By definition (2.4), the set $\mathcal{R} = \mathcal{R}^{(l)}$ consists now of functions u such that

$$||u||_{\mathcal{R}}^2 = \int_0^\infty (1 + x^{-2l}) |u(x)|^2 dx < \infty.$$

It follows from Lemma 3.4 that the form (3.11) is bounded on $\mathcal{R}^{(l)}$.

Lemma 3.9 *Let $p_k \in (-1/2-l, -1/2)$ for $k = 1, \dots, n$. Then the functionals $\Phi_{p_k}(u)$ defined by* (3.13) *are bounded on $\mathcal{R}^{(l)}$ and are strongly linear independent.*

Proof. – The inequality

$$|\Phi_p(u)| \le C||u||_{\mathcal{R}}$$

is equivalent to the inclusion

$$x^p(1 + x^{-2l})^{-1/2} \in L_2(\mathbb{R}_+)$$

which is true if $p \in (-1/2 - l, -1/2)$. The functionals $\Phi_{p_1}, \dots, \Phi_{p_n}$ are strongly linear independent because a function $\sum_{k=1}^n c_k x^{p_k}$ does not belong to the space $L_2(\mathbb{R}_+)$ unless all $c_k = 0$. $\square$

Put $b_l^{(\kappa)}(t) = \chi^{(\kappa)}(t) b_l(t)$. With the help of Lemmas 3.1, 3.6 and 3.7 it easy to show that

$$\int_0^\infty \int_0^\infty b_l^{(\kappa)}(xy)u(y)\overline{u(x)}dxdy = \int_{-\infty}^\infty \beta_l^{(\kappa)}(\lambda)(\mathbf{M}u)(-\lambda)\overline{(\mathbf{M}u)(\lambda)}d\lambda. \tag{3.14}$$

The precise statements are formulated in the two following assertions.

Lemma 3.10 *Let Assumption* 3.3 *hold and $u \in C_0^\infty(\mathbb{R}_+)$. Then representation* (3.14) *is valid for $\kappa = 1$.*

Lemma 3.11 *Let Assumption* 3.5 *hold and $u \in C_0^\infty(\mathbb{R}_+)$. Suppose that $l \ne r_n + 1/2$ for $n = 1, \dots, N$. Then representation* (3.14) *is valid for $\kappa = 0$.*

To check the part 1^0 of Theorem 3.8, we compare Lemmas 3.10, 3.11 and take into account equality (3.5). This yields the representation

$$b_l[u, u] = (\mathcal{B}_l U\mathbf{M}u, U\mathbf{M}u)_{L_2(\mathbb{R}_+;\mathbb{C}^2)}, \tag{3.15}$$

where $\mathcal{B}_l$ is multiplication by matrix (3.4) with elements (3.10). It follows from (3.12) and (3.15) that for all $u \in C_0^\infty(\mathbb{R}_+)$

$$a_l[u, u] = (A_l u, u) + \gamma \sum_{r_k < l-1/2} v_k |\Phi_{r_k - l}(u)|^2, \tag{3.16}$$

where

$$A_l = \gamma(U\mathbf{M})^* \mathcal{B}_l U\mathbf{M} \tag{3.17}$$

is a bounded operator in $\mathcal{H}$. Since, by Lemma 3.9, the functionals $\Phi_{r_k - l}(u)$ are bounded on $\mathcal{R}^{(l)}$, equality (3.16) extends by continuity to all $u \in \mathcal{R}^{(l)}$. This gives us representation (2.11) where $m = N_l^{(+)}$, $n - m = N_l^{(-)}$ if $\gamma > 0$ and $m = N_l^{(-)}$, $n - m = N_l^{(+)}$ if $\gamma < 0$. According to Lemma 3.9 the corresponding functionals $\varphi_1, \dots, \varphi_n$ are strongly linear independent. By virtue of Lemma 3.2, the total multiplicity of the spectrum of operator (3.17) in the interval $(-\infty, -1)$ is zero if $|\gamma| p_l \le 1$ and it is infinite if $|\gamma| p_l > 1$. Thus the assertion of the part 1^0 of Theorem 3.8 follows immediately from Theorem 2.12.

To check the part 2^0 of Theorem 3.8, we verify the assumptions of Theorems 2.4. We rely again on representation (3.12) valid at least for $u \in C_0^\infty(\mathbb{R}_+)$. Let $\mathcal{R}_0$ consist of functions $u \in C_0^\infty(\mathbb{R}_+)$ such that

$$\Phi_{-1/2}(u) = \int_0^\infty u(x) x^{-1/2} dx = 0. \tag{3.18}$$

Let us extend representation (3.14) for the form $b_l^{(0)}[u, u]$ to the case $l = r_n + 1/2$.

Lemma 3.12 *Let Assumption* 3.5 *hold and* $u \in \mathcal{R}_0$. *Then representation* (3.14) *is valid for* $\kappa = 0$ *and* $l = r_n + 1/2$ *with the function*

$$\beta^{(0)}_{r_n+1/2}(\lambda) = \int_0^1 \Big(v(t) - \sum_{k=1}^n v_k t^{r_k}\Big) t^{-1-r_n-i\lambda} dt + i v_n \lambda^{-1}. \tag{3.19}$$

Proof. – Let us proceed from equality (3.14) for $\kappa = 0$ and $l = r_n + 1/2 - \varepsilon$, $\varepsilon > 0$. Then we pass to the limit $\varepsilon \to 0$ in this equality. Its left-hand side is of course continuous in l for any $u \in C_0^\infty(\mathbb{R}_+)$. Let us verify the convergence of the right-hand side as $\varepsilon \to 0$. Comparing definition (3.8), (3.9) of the functions $\beta^{(0)}_l(\lambda)$ with (3.19), we see that

$$\beta^{(0)}_l(\lambda) = \int_0^1 \Big(v(t) - \sum_{k=1}^n v_k t^{r_k}\Big) t^{-1/2-l-i\lambda} dt + v_n (r_n + 1/2 - l - i\lambda)^{-1}.$$

These functions converge as $\varepsilon \to 0$ to function (3.19) uniformly in λ outside of any neighbourhood of the point $\lambda = 0$ and

$$|\beta^{(0)}_l(\lambda)| \leq C(|\lambda|^{-1} + 1).$$

This suffices to justify passing to the limit in the integral over λ because $\mathbf{M}u \in \mathcal{S}(\mathbb{R})$ and $(\mathbf{M}u)(0) = 0$ by virtue of condition (3.18). $\square$

Let now $l = r_n + 1/2$, $\beta_l(\lambda) = \beta^{(0)}_l(\lambda) + \beta^{(1)}_l(\lambda)$ where $\beta^{(0)}_l$ and $\beta^{(1)}_l$ are given by (3.19) and (3.9), respectively, and let $\mathcal{B}_l$ be multiplication by matrix (3.4) with the elements $\beta_l(\lambda)$. Lemmas 3.10 and 3.12 imply that, in the case $l = r_n + 1/2$, equality (3.15) holds for $u \in \mathcal{R}_0$. This gives us representation (3.16) where $u \in \mathcal{R}_0$ and the operator A_l is defined by equality (3.17). Since

$$\lim_{\lambda \to 0} |\lambda|\, |\beta_l(\lambda)| = |v_n| \neq 0, \quad l = r_n + 1/2, \tag{3.20}$$

it follows from Lemma 3.2 that the operator $\mathcal{B}_l$ is unbounded from below and consequently

$$\dim E_{A_l}(-\infty, -1) = \infty$$

for $l = r_n + 1/2$ and any $\gamma \neq 0$. Note also the following elementary

Lemma 3.13 *The set* $\mathcal{R}_0$ *is dense in* $\mathcal{D}(A_l)$ *where* $l = r_n + 1/2$.

Proof. – Recall that the function $\beta_l(\lambda)$ is bounded except the point $\lambda = 0$ where it satisfies (3.20). Therefore it follows from (3.4) and (3.17) that the inclusion $u \in \mathcal{D}(A_l)$ is equivalent to the bound

$$\int_{-\infty}^{\infty} (1 + \lambda^{-2}) |w(\lambda)|^2 d\lambda < \infty \quad \text{for} \quad w = \mathbf{M}u. \tag{3.21}$$

Clearly, the Mellin transform (3.1) can be factorized as $\mathbf{M} = \Phi G$ where Φ is the Fourier transform in $L_2(\mathbb{R})$ and $(Gu)(t) = e^{t/2} u(e^t)$. In terms of $g(\lambda) = \lambda^{-1} w(\lambda)$, (3.21) is equivalent to the condition

$$\int_{-\infty}^{\infty} (|\tilde{g}(t)|^2 + |\tilde{g}'(t)|^2) dt < \infty, \quad \tilde{g} = \Phi^* g. \tag{3.22}$$

Of course, there exists a sequence $\tilde{g}_j \in C_0^\infty(\mathbb{R})$ such that $\tilde{g}_j$ converge to $\tilde{g}$ as $j \to \infty$ in the metrics (3.22) (that is in the Sobolev space $H^1(\mathbb{R})$). Set $\tilde{w}_j(t) = -i\tilde{g}_j'(t)$ so that $w_j(\lambda) = \lambda g_j(\lambda)$. Then w_j converge to w in the metrics (3.21). Moreover, $\tilde{w}_j \in C_0^\infty(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \tilde{w}_j(t)dt = -i\int_{-\infty}^{\infty} \tilde{g}_j'(t)dt = 0.$$

It follows that $u_j = G^*\tilde{w}_j \in C_0^\infty(\mathbb{R}_+)$, u_j satisfy (3.18) and $u_j \to u$ in $\mathcal{D}(A_l)$ as $j \to \infty$. $\square$

Thus, we have verified all conditions of Theorem 2.4 and hence the negative spectrum of the operator H_γ is infinite for any $\gamma \neq 0$. This concludes the proof of Theorem 3.8.

4. The function (3.10) can be calculated on the basis of the following

Proposition 3.14 *Suppose that the integral*

$$\mathcal{V}(T) = \int_1^T v(t)t^{-1/2}dt$$

is bounded uniformly in $T \geq 1$ *and that Assumption* 3.5 *holds. Then the function*

$$\mathfrak{B}(z) = \int_0^\infty v(t)t^{-1/2-z}dt \tag{3.23}$$

is analytic in the band $\operatorname{Re} z \in (0, r_1 + 1/2)$ *and admits a meromorphic continuation in the band* $\operatorname{Re} z \in (0, r_{N+1} + 1/2)$*. The function* $\mathfrak{B}(z)$ *has only simple poles in the points* $r_n + 1/2$ *with the residues* $-v_n$*,* $n = 1, \dots, N$*. Moreover, it is given by the formula*

$$\mathfrak{B}(z) = \int_0^\infty \Big(v(t) - \sum_{k=1}^n v_k t^{r_k}\Big)t^{-1/2-z}dt \tag{3.24}$$

in the band $\operatorname{Re} z \in (r_n + 1/2, r_{n+1} + 1/2)$*.*

Proof. – Integrating by parts, we see that the integral

$$\mathfrak{B}_1(z) = \int_1^\infty v(t)t^{-1/2-z}dt = z\int_1^\infty \mathcal{V}(t)t^{-1-z}dt$$

defines an analytic function for all $\operatorname{Re} z > 0$. If $\operatorname{Re} z > a_n + 1/2$, then

$$\mathfrak{B}_1(z) = \int_1^\infty \Big(v(t) - \sum_{k=1}^n v_k t^{r_k}\Big)t^{-1/2-z}dt + \sum_{k=1}^n v_k (z - r_k - 1/2)^{-1}. \tag{3.25}$$

Similarly, if $\operatorname{Re} z < a_1 + 1/2$, then

$$\mathfrak{B}_0(z) = \int_0^1 v(t)t^{-1/2-z}dt = \int_0^1 \Big(v(t) - \sum_{k=1}^n v_k t^{r_k}\Big)t^{-1/2-z}dt - \sum_{k=1}^n v_k (z - r_k - 1/2)^{-1} \tag{3.26}$$

According to Assumption 3.5, the integral in the right-hand side of (3.26) is analytic for $\operatorname{Re} z < r_{n+1} + 1/2$ so that (3.26) gives the meromorphic continuation of the function $\mathfrak{B}_0(z)$. In particular, if $n = N$ we obtain that the function $\mathfrak{B}(z) = \mathfrak{B}_0(z) + \mathfrak{B}_1(z)$ is meromorphic in the band $\operatorname{Re} z \in (0, r_{N+1} + 1/2)$. Finally, comparing representations (3.25) and (3.26), we arrive at (3.24). $\square$

Thus, to calculate the function $\beta_l(\lambda)$, it suffices to compute integral (3.23) for $\operatorname{Re} z \in (0, r_1 + 1/2)$ and then to find its meromorphic continuation into the band $\operatorname{Re} z \in (0, r_{N+1} + 1/2)$. Putting together relations (3.8), (3.10) and (3.24), we see that

$$\beta_l(\lambda) = \mathfrak{B}(l + i\lambda), \quad l \neq r_n + 1/2. \tag{3.27}$$

4. EXAMPLES

1. Let us first consider the operator $H_\gamma = H_0 + \gamma V$ in the space $L_2(\mathbb{R}_+)$. Recall that H_0 is multiplication by x^{2l} and a perturbation V is defined by formula (1.2). As an example, we choose

$$v(t) = v_{p,q}(t) = t^q \mathcal{I}_p(t) \tag{4.1}$$

where $\mathcal{I}_p$ is the Bessel function. It follows from the asymptotics of $\mathcal{I}_p(t)$ at infinity and from its expansion at $t = 0$ that

$$v(t) = (2/\pi)^{1/2} t^{q-1/2}\Big(\cos(t-(2p+1)\pi/4) + (2t)^{-1}(4^{-1}-p^{-2})\sin(t-(2p+1)\pi/4)\Big) + O(t^{q-5/2})$$

as $t \to \infty$ and

$$v(t) = \sum_{k=0}^{\infty} (-1)^k 2^{-2k-p}\Big(k!\Gamma(k+p+1)\Big)^{-1} t^{p+q+2k}, \tag{4.2}$$

where $\Gamma(\cdot)$ is the Γ-function. Therefore the condition of Proposition 3.14 on the function $\mathcal{V}(T)$ is satisfied for all $q \le 1$ and (4.2) gives us relation (3.6) with numbers $r_k = p+q+2k$, where $k = 0, 1, 2, \ldots$. The corresponding coefficients v_k are positive for even k and are negative for odd k. So Assumption 3.5 is fulfilled for $p + q > -1/2$ and all $l > 0$. Using formula (19), section 7.7 of [1], vol. 2, we find that function (3.23) equals now

$$\mathfrak{B}(z) = 2^{q-z-1/2}\Gamma((p+q-z+1/2)/2)\Gamma^{-1}((p-q+z+3/2)/2). \tag{4.3}$$

Let us set

$$\sigma_l = 2^{-q+l+1/2} \min_{\lambda\in\mathbb{R}} |\Gamma((p-q+l+i\lambda+3/2)/2)\Gamma^{-1}((p+q-l-i\lambda+1/2)/2)|. \tag{4.4}$$

Remark that, by the Stirling formula, $\mathfrak{B}(l+i\lambda) \to 0$ as $|\lambda| \to \infty$ so that the spectrum of the corresponding operator $\mathcal{B}_l$ (see Lemma 3.2) consists of the interval $[-\sigma_l^{-1}, \sigma_l^{-1}]$. In our particular case, Theorem 3.8 gives the following assertion.

Proposition 4.1 *Let the function $v(t)$ be given by formula (4.1) where $-1/2 - p < q \le 1$. Then the negative spectrum of the operator H_γ is infinite for any $\gamma \ne 0$ if $l = p+q+1/2+2k$ for some $k = 0, 1, 2, \ldots$. In the opposite case it is infinite if $|\gamma| > \sigma_l$ where the number σ_l is defined by (4.4). If $\gamma \in [-\sigma_l, 0)$, then the negative spectrum of the operator H_γ is empty for $l < p+q+1/2$ and it consists of $k+1$ eigenvalues if $l \in (p+q+1/2+2k, p+q+5/2+2k)$. If $\gamma \in (0, \sigma_l]$, then the negative spectrum of the operator H_γ is empty for $l < p+q+5/2$ and it consists of $k+1$ eigenvalues if $l \in (p+q+5/2+2k, p+q+9/2+2k)$.*

We note special cases $p = 1/2$ when $q \in (-1, 1]$ and $p = -1/2$ when $q \in (0, 1]$:

$$v_{1/2,q}(t) = (2/\pi)^{1/2} t^{q-1/2} \sin t \quad \text{and} \quad v_{-1/2,q}(t) = (2/\pi)^{1/2} t^{q-1/2} \cos t. \tag{4.5}$$

2. Below we consider the operators $\mathbf{H}_\gamma^{(c)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(c)}$ and $\mathbf{H}_\gamma^{(s)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(s)}$ in the space $L_2(\mathbb{R}^d)$, where $\mathbf{H}_0$ is multiplication by $|x|^{2l}$ and $\mathbf{V}^{(c)}$, $\mathbf{V}^{(s)}$ are defined by formulas (1.1). Let $\mathfrak{h}_n$ be the subspace of spherical functions $Y_n(\omega)$, $\omega \in \mathbb{S}^{d-1}$, of order n, let $\mathcal{K}$ be the L_2-space

with weight r^{d-1} of functions defined on $\mathbb{R}_+$ and let $\mathfrak{H}_n = \mathcal{K} \otimes \mathfrak{h}_n$. To put it differently, $\mathfrak{H}_n \subset \mathbb{R}^d$ is the subspace of functions u_n of the form

$$u_n(x) = |x|^{-\delta} g(|x|) Y_n(\hat{x}), \quad \hat{x} = x|x|^{-1}, \quad \delta = (d-1)/2, \tag{4.6}$$

where $g \in L_2(\mathbb{R}_+)$ and $Y_n \in \mathfrak{h}_n$. Then

$$L_2(\mathbb{R}^d) = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n, \quad \mathfrak{H}_n = \mathcal{K} \otimes \mathfrak{h}_n,$$

and every subspace $\mathfrak{H}_n$ is invariant with respect to the Fourier operator Φ which reduces to the Fourier-Bessel transform on $\mathfrak{H}_n$. More precisely, set

$$(\Phi_n g)(r) = i^{-n} \int_0^\infty (rs)^{1/2} \mathcal{I}_{n+(d-2)/2}(rs) g(s) ds. \tag{4.7}$$

Then, for function (4.6),

$$(\Phi u_n)(x) = |x|^{-\delta} (\Phi_n g)(|x|) Y_n(\hat{x}).$$

The operator Φ_n is of course unitary on $L_2(\mathbb{R}_+)$. It follows from (4.6) that

$$(\mathbf{V}^{(c)} u_n)(x) = (-1)^{n/2} |x|^{-\delta} (\Phi_n g)(|x|) Y_n(\hat{x})$$

for even n and $\mathbf{V}^{(c)} u_n = 0$ for odd n. Similarly,

$$(\mathbf{V}^{(s)} u_n)(x) = (-1)^{(n+1)/2} |x|^{-\delta} (\Phi_n g)(|x|) Y_n(\hat{x})$$

for odd n and $\mathbf{V}^{(s)} u_n = 0$ for even n. Let us set $\tau_n = (-1)^{n/2}$ for even n, $\tau_n = (-1)^{(n+1)/2}$ for odd n,

$$H_\gamma^{(n)} = H_0 + \tau_n \gamma \Phi_n \tag{4.8}$$

and let $T : L_2(\mathbb{R}_+) \to \mathcal{K}$ be a unitary operator defined by $(Tg)(r) = r^{-\delta} g(r)$. Then

$$\mathbf{H}_\gamma^{(c)} = \bigoplus_{m=0}^{\infty} T H_\gamma^{(2m)} T^* \otimes I_{2m}, \quad \mathbf{H}_\gamma^{(s)} = \bigoplus_{m=0}^{\infty} T H_\gamma^{(2m+1)} T^* \otimes I_{2m+1}, \tag{4.9}$$

where I_n is the identity operator in the space $\mathfrak{h}_n$. Recall that

$$\dim \mathfrak{h}_n = (2n+d-2)(n+d-3)!((d-2)!n!)^{-1} =: \nu_n. \tag{4.10}$$

Comparing (4.1) and (4.7) and setting

$$p = n + (d-2)/2, \quad q = 1/2, \tag{4.11}$$

we see that Proposition 4.1 can be directly applied to every operator (4.8).

Proposition 4.2 *The negative spectrum of the operator $H_\gamma^{(n)}$ is infinite for any $\gamma \neq 0$ if $l = n + d/2 + 2k$ for some $k = 0, 1, 2, \ldots$. In the opposite case it is infinite if $|\gamma| > \sigma_l^{(n)}$ where*

$$\sigma_l^{(n)} = 2^l \min_{\lambda \in \mathbb{R}} |\Gamma((n + d/2 + l + i\lambda)/2) \Gamma^{-1}((n + d/2 - l - i\lambda)/2)|. \tag{4.12}$$

If $\tau_n \gamma \in [-\sigma_l^{(n)}, 0)$, then the negative spectrum of the operator $H_\gamma^{(n)}$ is empty for $l < d/2 + n$ and it consists of $k+1$ eigenvalues if $l \in (d/2 + n + 2k, d/2 + n + 2 + 2k)$. If $\tau_n \gamma \in (0, \sigma_l^{(n)}]$, then the negative spectrum of the operator $H_\gamma^{(n)}$ is empty for $l < d/2 + n + 2$ and it consists of $k+1$ eigenvalues if $l \in (d/2 + n + 2 + 2k, d/2 + n + 4 + 2k)$.

Combining Proposition 4.2 with decomposition (4.9) we can deduce results on the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$. Let us start with exceptional values of l.

Theorem 4.3 *Let $l = d/2 + 2k$ for some $k = 0, 1, 2, \dots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ is infinite for any $\gamma \neq 0$. Let $l = d/2 + 2k + 1$ for some $k = 0, 1, 2, \dots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(s)}$ is infinite for any $\gamma \neq 0$.*

3. To consider other values of l, we first find a relation between functions $\beta_l^{(n)}(\lambda)$ associated to different operators $H_\gamma^{(n)}$. Remark that according to (3.27), (4.3), in the case (4.11),

$$|\beta_l^{(n)}(\lambda)| = 2^{-l} |\Gamma((n + d/2 - l + i\lambda)/2)\Gamma^{-1}((n + d/2 + l + i\lambda)/2)|.$$

It follows from the identity

$$\Gamma(z + 1) = z\Gamma(z) \tag{4.13}$$

that

$$|\beta_l^{(n+2)}(\lambda)| = |n + d/2 - l + i\lambda||n + d/2 + l + i\lambda|^{-1} \, |\beta_l^{(n)}(\lambda)| \le |\beta_l^{(n)}(\lambda)|.$$

Therefore the numbers

$$\sigma_l^{(n)} = \min_{\lambda \in \mathbb{R}} |\beta_l^{(n)}(\lambda)|^{-1}$$

are related by the inequality $\sigma_l^{(n)} \le \sigma_l^{(n+2)}$. In particular, we obtain the following result.

Lemma 4.4 *For any $m = 0, 1, 2, \dots$*

$$\sigma_l^{(2m)} \ge \sigma_l^{(0)}, \quad \sigma_l^{(2m+1)} \ge \sigma_l^{(1)}.$$

Let us check that the minimum in definition (4.12) of $\sigma_l^{(0)}$ and $\sigma_l^{(1)}$ is attained at the point $\lambda = 0$. To that end we need the following assertion from the theory of the Γ-function.

Lemma 4.5 *Let $b > 0$, $a \le b$ and $\lambda \in \mathbb{R}$. Then inequality*

$$|\Gamma(a + i\lambda)\Gamma^{-1}(b + i\lambda)| \le |\Gamma(a)\Gamma^{-1}(b)| \tag{4.14}$$

holds in the following three cases: 1^0 $a > 0$, 2^0 $a = -n + \varepsilon$ where $n = 1, 2, \dots$, $\varepsilon \in (0, 1)$ and $\varepsilon \le b$, 3^0 $a \in (-1, 0)$ and $|a| \le b$.

Proof. – Let us start with the first case. Clearly, (4.14) is equivalent to the inequality

$$\Gamma(a + i\lambda)\Gamma(a - i\lambda)\Gamma(a)^{-2} \le \Gamma(b + i\lambda)\Gamma(b - i\lambda)\Gamma(b)^{-2}, \quad 0 < a \le b.$$

Thus, it suffices to check that for any $\lambda \in \mathbb{R}$ the derivative of the function

$$\varphi(a, \lambda) = \Gamma(a + i\lambda)\Gamma(a - i\lambda)\Gamma(a)^{-2}$$

with respect to a is nonnegative. Calculating this derivative and denoting $\psi(z) = \Gamma(z)^{-1}\Gamma'(z)$, we find that

$$\varphi(a, \lambda)^{-1}\partial\varphi(a, \lambda)/\partial a = \psi(a + i\lambda) + \psi(a - i\lambda) - 2\psi(a).$$

It follows from the Dirichlet representation (formula (20), section 1.7 of [1], vol. 1)

$$\psi(z) = \int_0^\infty (e^{-x} - (1+x)^{-z})x^{-1}dx, \quad \mathrm{Re}\, z > 0,$$

that

$$\psi(a+i\lambda) + \psi(a-i\lambda) - 2\psi(a) = 2\int_0^\infty (1-\cos(\lambda \ln(1+x)))(1+x)^{-a}x^{-1}dx \geq 0, \quad a > 0.$$

To consider the case 2^0, we remark that, by (4.13)

$$|\Gamma(a+i\lambda)\Gamma^{-1}(b+i\lambda)| = |(-n+\varepsilon+i\lambda)^{-1}\cdots(-1+\varepsilon+i\lambda)^{-1}\Gamma(\varepsilon+i\lambda)\Gamma^{-1}(b+i\lambda)|. \quad (4.15)$$

Using (4.14) for the numbers ε (in place of a), b and obvious estimates $|(-k+\varepsilon+i\lambda)^{-1}| \leq |(-k+\varepsilon)^{-1}|,\ k = 1, \cdots, n$, we find that the right-hand side of (4.15) is bounded by

$$|(-n+\varepsilon)^{-1}\cdots(-1+\varepsilon)^{-1}\Gamma(\varepsilon)\Gamma^{-1}(b)|,$$

which, again by (4.13), equals $|\Gamma(a)\Gamma^{-1}(b)|$.

To prove the part 3^0, we use again (4.13), apply inequality (4.14) to the numbers $a+1$ and $b+1$ and remark that

$$|b+i\lambda||a+i\lambda|^{-1} \leq b|a|^{-1}.$$

This yields

$$|\Gamma(a+i\lambda)\Gamma^{-1}(b+i\lambda)| \leq |\Gamma(a+1+i\lambda)\Gamma^{-1}(b+1+i\lambda)||b+i\lambda||a+i\lambda|^{-1} \leq |\Gamma(a+1)\Gamma^{-1}(b+1)|b|a|^{-1}.$$

The right-hand side here equals $|\Gamma(a)\Gamma^{-1}(b)|$. $\square$

Now we can simplify expressions for $\sigma_l^{(0)}$ and $\sigma_l^{(1)}$.

Lemma 4.6 *Put*

$$\sigma_l^{(c)} = 2^l|\Gamma((d/2+l)/2)\Gamma^{-1}((d/2-l)/2)|, \quad \sigma_l^{(s)} = 2^l|\Gamma((d/2+l+1)/2)\Gamma^{-1}((d/2-l+1)/2)|. \quad (4.16)$$

Then $\sigma_l^{(0)} = \sigma_l^{(c)}$ *and* $\sigma_l^{(1)} = \sigma_l^{(s)}$.

Proof. – Consider first $\sigma_l^{(1)}$. According to (4.14) it suffices to check that numbers $a = (d/2-l+1)/2$ and $b = (d/2+l+1)/2$ satisfy one of the three conditions of Lemma 4.5. If $d \geq 2$, then $b > 1$ and hence condition 2^0 holds for all $l > 0$. If $d = 1$, we distinguish the cases $l > 3/2$ and $l < 3/2$. In the first of them $b > 1$ so that condition 2^0 is fulfilled and in the second $a > 0$ so that condition 1^0 is fulfilled.

Similarly, to consider $\sigma_l^{(0)}$, we need to check that numbers $a = (d/2-l)/2$, $b = (d/2+l)/2$ also satisfy one of the three conditions of Lemma 4.5. If $d \geq 4$, then $b > 1$ and hence condition 2^0 holds for all $l > 0$. If $d = 3$, we distinguish the cases $l > 3/2$ and $l < 3/2$. In the first of them $b > 1$ so that condition 2^0 is fulfilled and in the second $a > 0$ so that condition 1^0 is fulfilled. If $d = 2$, we distinguish the cases $l > 1$ and $l < 1$. In the first of them $b > 1$ and condition 2^0 is fulfilled. In the second $a > 0$ and condition 1^0 is fulfilled. Let, finally, $d = 1$. If $l < 1/2$, then $a = 1/4 - l/2 > 0$ so that condition 1^0 holds. If $l > 3/2$, then

$b = 1/4 + l/2 > 1$ so that condition 2^0 holds. Finally, in the case $l \in (1/2, 3/2)$ we have that $a \in (-1/2, 0)$ and $|a| = l/2 - 1/4$. Therefore $|a| < b$ and we can refer to condition 3^0. $\square$

4. Let us return to the operators $\mathbf{H}_\gamma^{(c)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(c)}$ and $\mathbf{H}_\gamma^{(s)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(s)}$ in the space $L_2(\mathbb{R}^d)$. We consider first the one-dimensional case $d = 1$ when (4.9) reduces to the decomposition of the space $L_2(\mathbb{R})$ into the subspaces of the even and odd functions. These subspaces are invariant with respect to $\mathbf{H}_\gamma^{(c)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(c)}$ and $\mathbf{H}_\gamma^{(s)} = \mathbf{H}_0 + \gamma \mathbf{V}^{(s)}$, $\mathbf{V}^{(c)} f = 0$ for odd f and $\mathbf{V}^{(s)} f = 0$ for even f. Therefore, the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ (respectively, $\mathbf{H}_\gamma^{(s)}$) in the space $L_2(\mathbb{R})$ coincides with that of the operator H_γ for $v(t) = (2/\pi)^{1/2} \cos t$ (respectively, $v(t) = (2/\pi)^{1/2} \sin t$) in the space $L_2(\mathbb{R}_+)$. Thus, we can directly apply Proposition 4.1, where according to (4.5) $p = -1/2, q = 1/2$ for the operator $\mathbf{H}_\gamma^{(+)}$ and $p = 1/2, q = 1/2$ for the operator $\mathbf{H}_\gamma^{(-)}$. Moreover, in the case $d = 1$ expressions (4.16) can be a little bit simplified. This gives us the following result.

Theorem 4.7 *Let $d = 1$. Put*

$$\sigma_l^{(c)} = (\pi/2)^{1/2} |\cos(\pi(1/2 - l)/2)\Gamma(1/2 - l)|^{-1}, \quad \sigma_l^{(s)} = (\pi/2)^{1/2} |\sin(\pi(1/2 - l)/2)\Gamma(1/2 - l)|^{-1}.$$

Suppose that $l \neq 1/2 + 2k$ for any $k = 0, 1, 2, \ldots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ is empty if $l \in (0, 1/2)$ and $|\gamma| \leq \sigma_l^{(c)}$. If $l \in (2k + 1/2, 2k + 5/2)$, then it consists of $[(k+1)/2]$ eigenvalues for $\gamma \in (0, \sigma_l^{(c)}]$ and it consists of $[k/2] + 1$ eigenvalues for $\gamma \in [-\sigma_l^{(c)}, 0)$. If $|\gamma| \geq \sigma_l^{(c)}$, then, for any l, the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ is infinite.

Suppose that $l \neq 3/2 + 2k$ for any $k = 0, 1, 2, \ldots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(s)}$ is empty if $l \in (0, 3/2)$ and $|\gamma| \leq \sigma_l^{(s)}$. If $l \in (2k + 3/2, 2k + 7/2)$, then it consists of $[(k+1)/2]$ eigenvalues for $\gamma \in (0, \sigma_l^{(s)}]$ and it consists of $[k/2] + 1$ eigenvalues for $\gamma \in [-\sigma_l^{(s)}, 0)$. If $|\gamma| \geq \sigma_l^{(s)}$, then, for any l, the negative spectrum of the operator $\mathbf{H}_\gamma^{(s)}$ is infinite.

5. Let now $d \geq 2$. It follows from Proposition 4.2 and Lemmas 4.4, 4.6 that in the case $|\gamma| \leq \sigma_l^{(c)}$ (respectively, $|\gamma| \leq \sigma_l^{(s)}$) the negative spectra of all operators $H_\gamma^{(2m)}$ (respectively, $H_\gamma^{(2m+1)}$) are finite. Moreover, they are empty for m large enough. Therefore, by virtue of (4.9), in this case the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ (respectively, $\mathbf{H}_\gamma^{(s)}$) is finite. On the other hand, if $|\gamma| > \sigma_l^{(c)}$ (respectively, $|\gamma| > \sigma_l^{(s)}$), then the negative spectrum of the operator $H_\gamma^{(0)}$ (respectively, $H_\gamma^{(1)}$) is infinite. This gives us necessary and sufficient conditions of the finiteness of the negative spectum of these operators.

Theorem 4.8 *Let the numbers $\sigma_l^{(c)}$ and $\sigma_l^{(s)}$ be defined by* (4.16). *Suppose that $l \neq d/2 + 2k$ for any $k = 0, 1, 2, \ldots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(c)}$ is finite if and only if $|\gamma| \leq \sigma_l^{(c)}$. Suppose that $l \neq d/2 + 2k + 1$ for any $k = 0, 1, 2, \ldots$. Then the negative spectrum of the operator $\mathbf{H}_\gamma^{(s)}$ is finite if and only if $|\gamma| \leq \sigma_l^{(s)}$.*

Note that Theorems 4.3 and 4.8 can be unified since $\sigma_l^{(c)} = 0$ (respectively, $\sigma_l^{(s)} = 0$) if $l = d/2 + 2k$ (respectively, $l = d/2 + 2k + 1$) for some $k = 0, 1, 2, \ldots$.

It remains to calculate the total numbers $\mathbf{N}_{l,c}^{(\pm)}$ and $\mathbf{N}_{l,s}^{(\pm)}$ of negative eigenvalues of the operators $\mathbf{H}_\gamma^{(c)}$ and $\mathbf{H}_\gamma^{(s)}$ in the cases $\pm\gamma \in (0, \sigma_l^{(c)}]$ and $\pm\gamma \in (0, \sigma_l^{(s)}]$, respectively. We

proceed from decomposition (4.9) and rely on Proposition 4.2. Recall also that numbers ν_l were defined by equality (4.10). Consider, for example, $\mathbf{H}_\gamma^{(c)}$. Suppose first that $\gamma \in [-\sigma_l^{(c)}, 0)$. If $l < d/2$, then $H_\gamma^{(2m)} \geq 0$ for all m so that $\mathbf{N}_{l,c}^{(-)} = 0$. If $l \in (d/2, d/2+2)$, then the operator $H_\gamma^{(0)}$ has one negative eigenvalue and $H_\gamma^{(2m)} \geq 0$ for $m \geq 1$. Since $\nu_0 = 1$, in this case $\mathbf{N}_{l,c}^{(-)} = 1$. If $l \in (d/2+2, d/2+4)$, then the operator $H_\gamma^{(0)}$ has two negative eigenvalues and $H_\gamma^{(2m)} \geq 0$ for $m \geq 1$ so that $\mathbf{N}_{l,c}^{(-)} = 2$. If $l \in (d/2+4, d/2+6)$, then the operator $H_\gamma^{(0)}$ has three negative eigenvalues, the operators $H_\gamma^{(2)}$ and $H_\gamma^{(4)}$ have one negative eigenvalue each and $H_\gamma^{(2m)} \geq 0$ for $m \geq 3$. It follows that in this case $\mathbf{N}_{l,c}^{(-)} = 3 + \nu_2 + \nu_4$. Repeating this procedure, we arrive at the general formula for the case $l \in (d/2+2k, d/2+2k+2)$:

$$\mathbf{N}_{l,c}^{(-)} = k + 1 + \sum_{p=0}^{[(k-1)/2]} (k - 2p - 1)(\nu_{4p+2} + \nu_{4p+4}), \quad k \geq 1. \tag{4.17}$$

The case $\gamma \in (0, \sigma_l^{(c)}]$ can be studied quite similarly. Now $H_\gamma^{(2m)} \geq 0$ for all m if $l < d/2+2$ and hence $\mathbf{N}_{l,c}^{(+)} = 0$ for such l. If $l \in (d/2+2, d/2+4)$, then the operators $H_\gamma^{(0)}$ and $H_\gamma^{(2)}$ have one negative eigenvalue each and $H_\gamma^{(2m)} \geq 0$ for $m \geq 2$. It follows that in this case $\mathbf{N}_{l,c}^{(+)} = \nu_0 + \nu_2$. If $l \in (d/2+4, d/2+6)$, then the operators $H_\gamma^{(0)}$ and $H_\gamma^{(2)}$ have two negative eigenvalues each and again $H_\gamma^{(2m)} \geq 0$ for $m \geq 2$ so that $\mathbf{N}_{l,c}^{(+)} = 2(\nu_0 + \nu_2)$. If $l \in (d/2+6, d/2+8)$, then the operators $H_\gamma^{(0)}$ and $H_\gamma^{(2)}$ have three negative eigenvalues each, both operators $H_\gamma^{(4)}$ and $H_\gamma^{(6)}$ have exactly one negative eigenvalue and $H_\gamma^{(2m)} \geq 0$ for $m \geq 4$. In this case $\mathbf{N}_{l,c}^{(+)} = 3(\nu_0 + \nu_2) + (\nu_4 + \nu_6)$. The general formula for the case $l \in (d/2+2k, d/2+2k+2)$ reads as

$$\mathbf{N}_{l,c}^{(+)} = \sum_{p=0}^{[k/2]} (k - 2p)(\nu_{4p} + \nu_{4p+2}). \tag{4.18}$$

Let us formulate these results.

Theorem 4.9 *The number $\mathbf{N}_{l,c}^{(-)} = 0$ for $l < d/2$, $\mathbf{N}_{l,c}^{(-)} = 1$ for $l \in (d/2, d/2+2)$ and $\mathbf{N}_{l,c}^{(-)}$ is determined by formula (4.17) for $l \in (d/2+2k, d/2+2k+2)$. The number $\mathbf{N}_{l,c}^{(+)} = 0$ for $l < d/2+2$ and $\mathbf{N}_{l,c}^{(+)}$ is determined by formula (4.18) for $l \in (d/2+2k, d/2+2k+2)$.*

The total numbers $\mathbf{N}_{l,s}^{(\pm)}$ of negative eigenvalues of the operator $\mathbf{H}_\gamma^{(s)}$ can be found quite similarly.

Theorem 4.10 *The number $\mathbf{N}_{l,s}^{(+)} = 0$ for $l < d/2+1$, $\mathbf{N}_{l,s}^{(+)} = \nu_1$ for $l \in (d/2+1, d/2+3)$ and*

$$\mathbf{N}_{l,s}^{(+)} = (k+1)\nu_1 + \sum_{p=1}^{[(k+1)/2]} (k - 2p + 1)(\nu_{4p-1} + \nu_{4p+1})$$

for $l \in (d/2+1+2k, d/2+3+2k)$, $k \geq 1$. The number $\mathbf{N}_{l,s}^{(-)} = 0$ for $l < d/2+3$ and

$$\mathbf{N}_{l,s}^{(-)} = \sum_{p=0}^{[k/2]} (k - 2p)(\nu_{4p+1} + \nu_{4p+3}).$$

for $l \in (d/2+1+2k, d/2+3+2k)$.

We emphasize that the numbers $\mathbf{N}_{l,c}^{(\pm)}$ and $\mathbf{N}_{l,s}^{(\pm)}$ do not depend on coupling constants $\pm\gamma \in (0, \sigma_l^{(c)}]$ and $\pm\gamma \in (0, \sigma_l^{(s)}]$, respectively.

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